Exercise of Section 1.1

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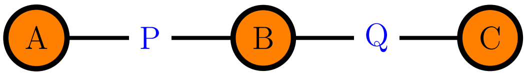
1. Since the vertices in the same partite set is independent, then if a complete bipartite graph $K_{p,q}$ is complete graph, it must be $K_{1,1}$, i.e. it is K_2.

2. 

adjacency matrix of P_3

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} A & C & B \end{array} \\ \begin{array}{c} A \\ C \\ B \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} B & C & A \end{array} \\ \begin{array}{c} B \\ C \\ A \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array},$$

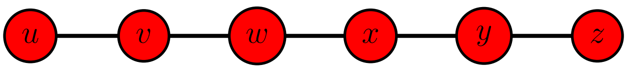
$$\begin{array}{cc} & \begin{array}{ccc} B & A & C \end{array} \\ \begin{array}{c} B \\ A \\ C \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} C & B & A \end{array} \\ \begin{array}{c} C \\ B \\ A \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} C & A & B \end{array} \\ \begin{array}{c} C \\ A \\ B \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

incidence matrix of P_3

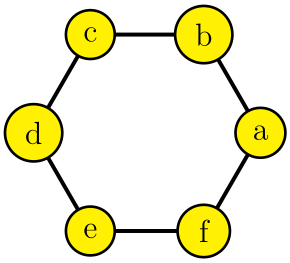
$$\begin{array}{cc} & \begin{array}{cc} P & Q \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} P & Q \end{array} \\ \begin{array}{c} A \\ C \\ B \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} P & Q \end{array} \\ \begin{array}{c} B \\ C \\ A \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} P & Q \end{array} \\ \begin{array}{c} B \\ A \\ C \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} P & Q \end{array} \\ \begin{array}{c} C \\ B \\ A \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} P & Q \end{array} \\ \begin{array}{c} C \\ A \\ B \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} Q & P \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} Q & P \end{array} \\ \begin{array}{c} A \\ C \\ B \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} Q & P \end{array} \\ \begin{array}{c} B \\ C \\ A \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} Q & P \end{array} \\ \begin{array}{c} B \\ A \\ C \end{array} & \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} Q & P \end{array} \\ \begin{array}{c} C \\ B \\ A \end{array} & \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} Q & P \end{array} \\ \begin{array}{c} C \\ A \\ B \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$



adjacency matrix of P_6 $$\begin{array}{cc} & \begin{array}{cccccc} u & v & w & x & y & z \end{array} \\ \begin{array}{c} u \\ v \\ w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$



adjacency matrix of C_6

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

3. $\begin{pmatrix} O_{n,m} & E_n \\ E_m & O_{m,n} \end{pmatrix}$ where E_n is the n-by-n matrix in which every entry is 1 and $O_{n,m}$ is the n-by-m matrix in which every entry is 0.

4. Since The complement of $\overline{G}$ is G, so we only need to prove Necessity Condition.

   Necessity. If $G \cong H$, then there is a bijection $f : V(G) \to V(H)$ such that

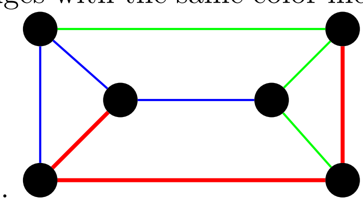
$$\begin{aligned} & uv \in E(G) \text{ if and only if } f(u)f(v) \in E(H) \\ \Longrightarrow & uv \notin E(G) \text{ if and only if } f(u)f(v) \notin E(H) \\ \Longrightarrow & uv \in E(\overline{G}) \text{ if and only if } f(u)f(v) \in E(\overline{H}) \end{aligned}$$

   Thus $\overline{G} \cong \overline{H}$.

5. The answer is **NO**. Consider a disconnected graph G which each component is cycle. Thus G is not a cycle.

   Note: The correct statement should be "If every vertex of a simple **connected** graph G has degree 2, then G is a cycle"

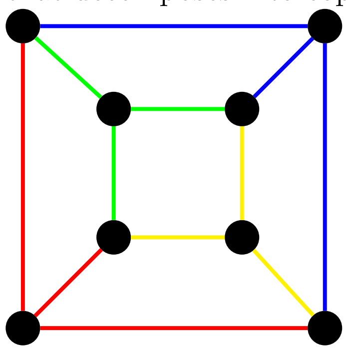
6. Since the edges with the same color induces P_4, thus the graph decomposes into copies of P_4.

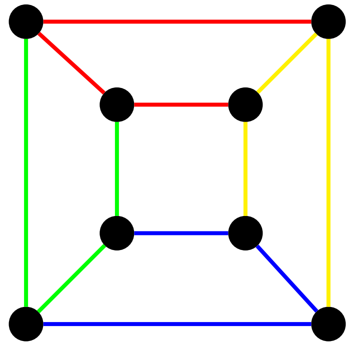


7. Let the graph be G and decomposes into path P_1, P_2 and P_3. For each vertex v, $deg_G(v) = deg_{P_1}(v) + deg_{P_2}(v) + deg_{P_3}(v)$. Since there are only two vertices of

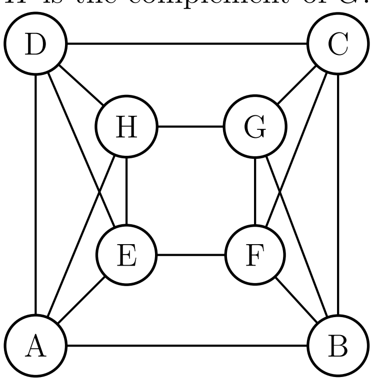
degree odd (in fact, it is 1) in each path P_i, then there are at most six vertices of odd degree in G.

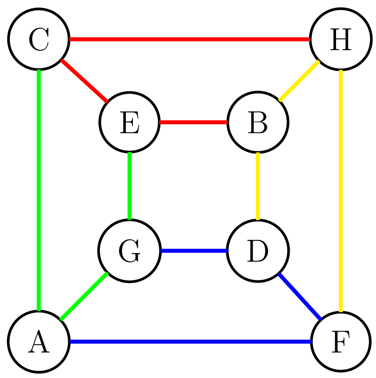
8. The graph left below is that decomposes into copies of $K_{1,3}$ and the other is that decomposes into copies of P_4.





9. Let the graph on the left below be G and the other be H. It's easy to check the H is the complement of G. Thus the proof is desired.



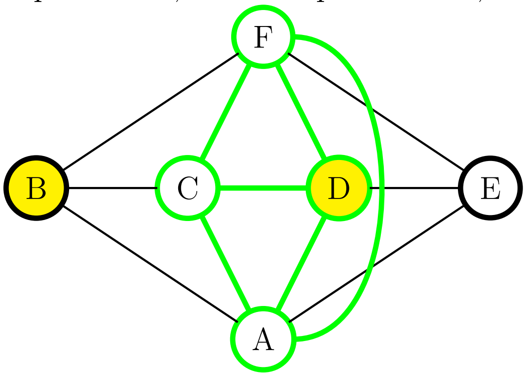


10. Let the simple disconnected graph be G, if v and u are in different components of G, then v is adjacent to u in $\overline{G}$. Otherwise, there exists a vertex w such that w and v in the different components, then u, w, v induced a path. So $\overline{G}$ is connected.

    Note: Suppose $G - v$ has r components, then $\overline{G} - v$ contains an induced subgraph H isomorphic to complete r-partite graph, where $n(H) = n(G)$.

11. Label the vertices as below, since A and F are adjacent to every vertex, then A and F must be in the maximum clique and they are not belong to any inde-

pendent set. We know B, C, D, E induces P_4, then there are at most 2 vertices in the maximum clique, and at most 2 vertices in the maximum independent set. Hence the size of maximum clique and maximum independent set is 4 and 2. It's easy to see that A, C, F, D is clique and B, D is independent set, thus



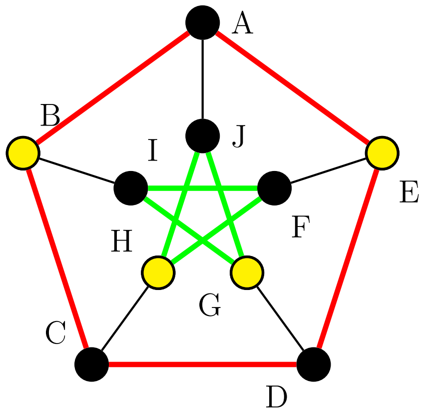
the maximum size is exactly 4 and 2.

Note: When I write $a_1, \ldots, a_k$ induced a path P_k that means a_i is adjacent to a_{i+1} for $i = 1, \ldots, k-1$.

12. If it is bipartite, then $\{A, C\}$ and $\{B, D\}$ must be in different partite sets, say PS_1 and PS_2. But E is adjacent to A and D, then $E \notin PS_1$ and $E \notin PS_2$. Thus the Petersen Graph is not bipartite.

    Note: By *Theorem 1.2.18*, Petersen Graph has odd cycle (see the graph below), then it is not bipartite.

    Since Petersen Graph contain two C_5 and it's easy to see that there are at most 2 independent vertices in C_5, thus there are at most 4 independent vertices and



it is easy to see $\{B, E, H, G\}$ is independent.

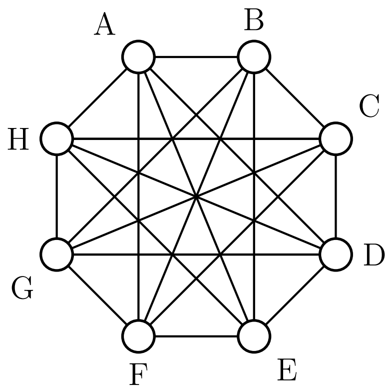
13. For all k, let $O = (\underbrace{0, \ldots\ldots\ldots, 0}_{k})$. Let $B_1 = \{v | v$ and O differ in exactly odd position$\}$ and $B_2 = \{v | v$ and O differ in exactly even position$\}$. By definition, B_1 and B_2 are both independent, thus G is bipartite.

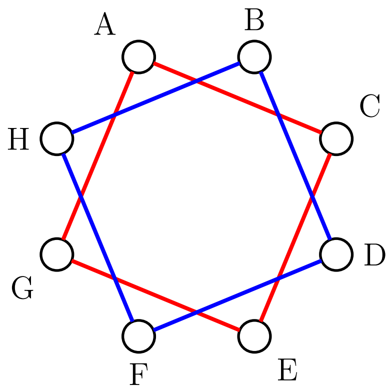
14. Label each square of 8-by-8 checkerboard as (i, j), where $1 \leq i, j \leq 8$. Set (i, j) be black if $i + j$ is odd and set it white if $i + j$ is even. And set 1-by-2 and 2-by-1 rectangle as one white and black. We remove $(0, 0)$ and $(8, 8)$, then there 30 white squares and 32 black squares, so it can not be partitioned into 1-by-2 and 2-by-1 rectangles. If a bipartite graph with different size of partite sets, then it contains no **perefect matching**.

    Note: The definition of perfect matching is *Definition 3.1.1*

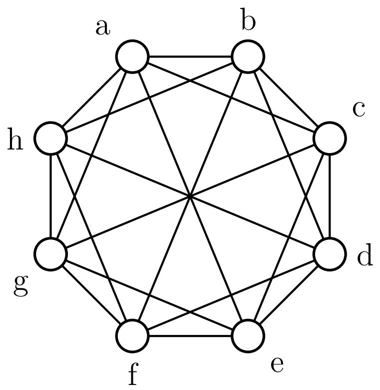
15. $A \bigcap B = \emptyset$; $A \bigcap C = \{K_2\}$; $A \bigcap D = \{K_2\}$; $B \bigcap C = \{K_3\}$; $B \bigcap D = \{C_n | n$ is even$\}$; $C \bigcap D = \emptyset$;

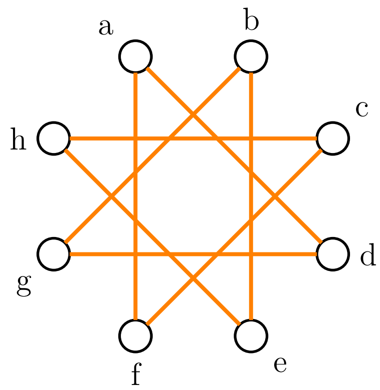
16. Let the graph left below is the left graph G and the other is $\overline{G}$.





Let the graph left below is the right graph H and the other is $\overline{H}$.

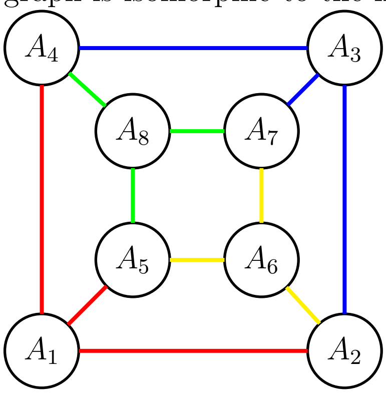


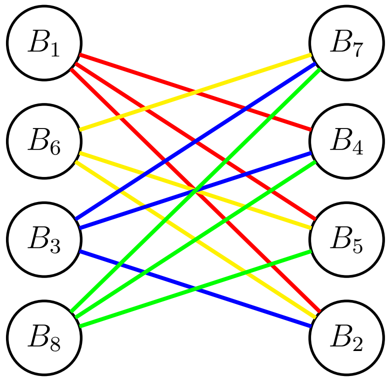


Thus it is easy to see $\overline{G}$ is $2C_4$ and $\overline{H}$ is C_8, then $\overline{G} \ncong \overline{H}$. By Exercise 1.1.4, we have $G \ncong H$.

17. Let the simple 7-vertex graph with the property be G, thus every vertex of $\overline{G}$ has degree 2. Thus by Exercise 1.1.5, we have $\overline{G}$ is $C_3 + C_4$ or C_7. Since $C_3 + C_4$ and C_7 are not isomorphic, then there are exactly 2 isomorphism classes.

18. It is easy to see the bijection function $f : \{A_i\} \to \{B_i\}$ is isomorphic, then left graph is isomorphic to the middle.



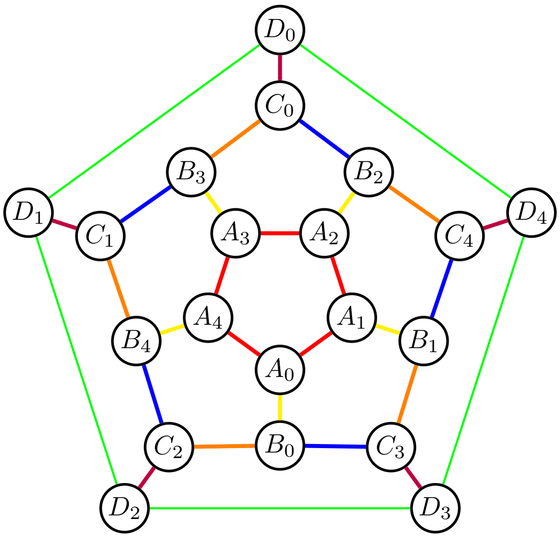


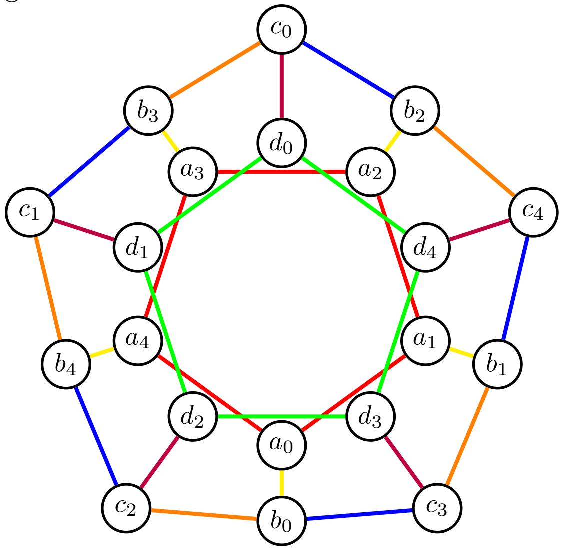
Since each two vertex in the right graph either adjacent or have common neighbor, but non-adjacent vertices A_1 and A_7 has no common neighbors.

Note : After studying **diameter** (*Definition 2.1.9*), we cae use the diameter of right graph is 2 but the diameter of the left graph is 3, so they are not isomorphic.

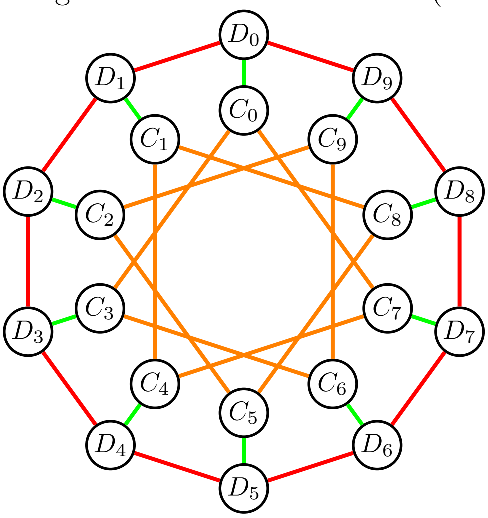
19. It is easy to see the function $f : \{A_i, B_i, C_i, D_i\} \to \{a_i, b_i, c_i, d_i\}$ is isomorphism,

then middle graph is isomorphic to the right.

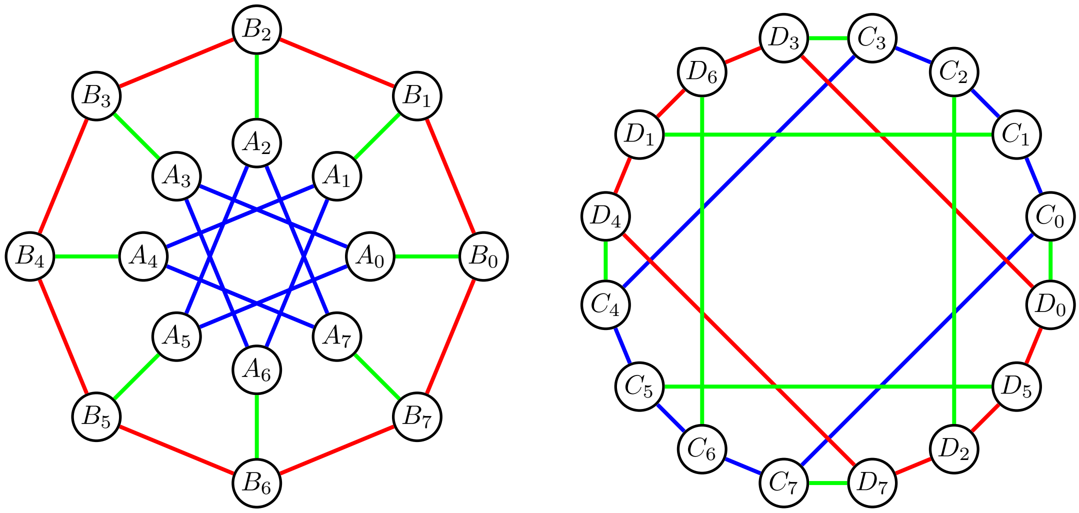




The graph below is the left graph. If a cycle has no orange(red) edges, then it is 10-cycle. Hence a cylce C which is smaller than 10-cycle, then C must have 2 green edges, 1 red edge and 1 orange edge. If it contains only one orange edge, say C_0C_3, then C_0C_3 belongs a 6-cycle or 10-cycle. If it contains only two orange edges, say C_0C_3 and C_3C_6, then they belong a 8-cycle or 10-cycle. Otherwise, C contains at least 6 edges. So the girth of left graph is 6. However the girth of middle is at most 5(in fact, exactly 5), so they are not isomorphic.

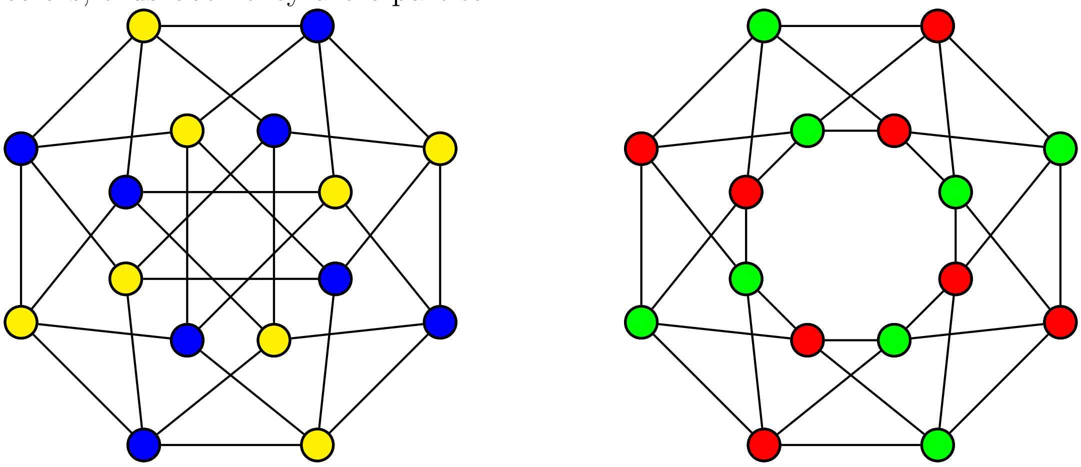


20. It is easy to see the function $f : \{A_i, B_i\} \to \{C_i, D_i\}$ is isomorphism, then left graph is isomorphic to the right.

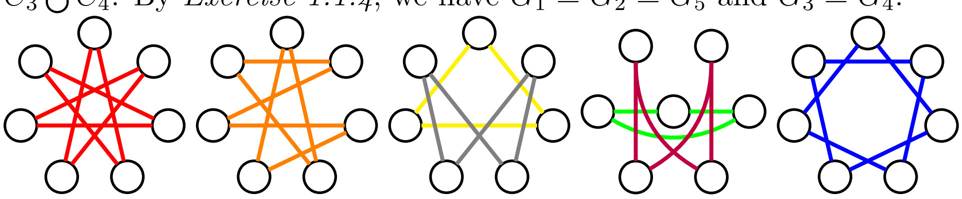


Similarly discussion as *Exercise 1.1.19*, we can get the girth of left graph is 6 but the girth of middle is 4, so they are not isomorphic.

21. Since the vertices with the same color induced independent set and use two colors, thus both they are bipartite.



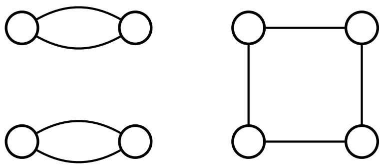
22. Let the graphs be $G_1, \ldots, G_5$ from left to right, thus consider $\overline{G_1}, \ldots, \overline{G_5}$. Clearly that $\overline{G_1}$, $\overline{G_2}$, $\overline{G_5}$ are isomorphic to C_7 and $\overline{G_3}$, $\overline{G_4}$ are isomorphic to $C_3 \bigcup C_4$. By *Exercise 1.1.4*, we have $G_1 \cong G_2 \cong G_5$ and $G_3 \cong G_4$.



23. (a) Obviously that $n > 1$, so in all graph, consider the graphs below. Both their degree sequence is $\{2, 2\}$, but the left is disconnect, the right one is

connected. So we have the minimum is 2.

(b) It is easy to see that for all loopless 2-vertex graph with the same degree sequence are isomorphic. For 3-vertex graph, may assume it is connected and let the degree sequence $\{a, b, c\}$ with vertex $\{A, B, C\}$. Thus easy to calculate the number of edge(BC), edge(AC) and edge(AB) are $(b+c-a)/2$, $(a+c-b)/2$ and $(a+b-c)/2$. Hence this the only one way to drawing 3-vertex graph. Now consider the graphs below. Both their degree sequence is $\{2, 2, 2, 2\}$, but the left is disconnect, the right one is connected. So we

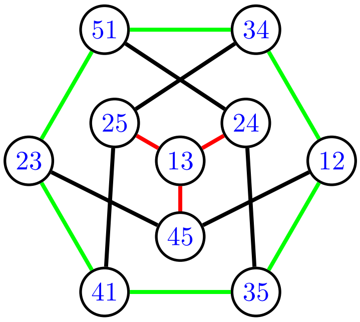


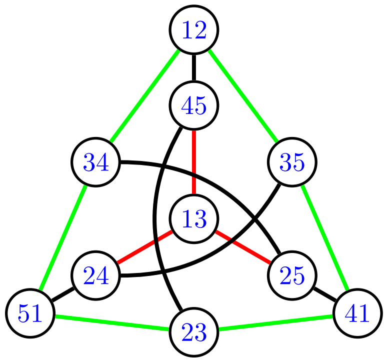
have the minimum is 4.

(c)

24. First of all, we have to know Petersen graph has 5-cycle, 6-cycle, 8-cycle and 9-cycle but no 7-cycle(by *Exercise 1.1.25*).

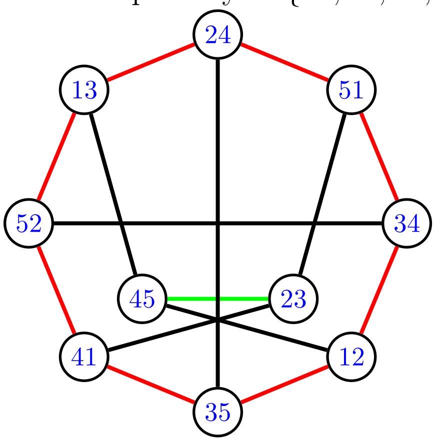
Consider 6-cycle of Petersen graph induced by $v_0, \ldots, v_5$, by *Proposition 1.1.38* we know for $i = 0, 1, 2$ v_i and v_{i+3} have the common neighbor u_i. But for each u_i, they are not in the 6-cycle, thus there is only one vertex w which is not adjacent to this 6-cycle and w, u_0, u_1, u_2 induced a claw. So we can draw Petersen graph by putting the claw in the center of 6-cycle. For example 6-cycle $\{12, 34, 51, 23, 41, 35\}$, and claw $\{31, 45, 24, 25\}$.

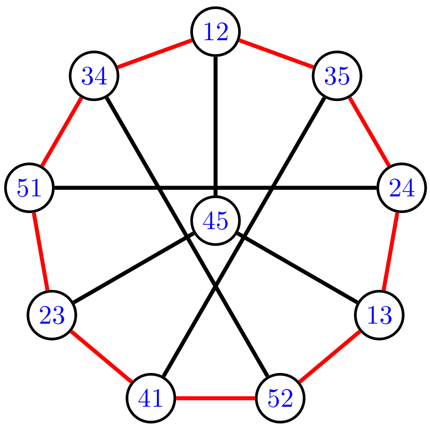




Next consider the 8-cycle induced by $\{v_0, \dots v_7\}$ and the other two vertices as w_1, w_2. Since Petersen graph is 3-regular, then by *Proposition 1.1.38* and *Corollary 1.1.40* we have $v_i, v_{i+2}, v_{i+4}, v_{i+6}$, for $i = 0, 1$. So we may assume v_0, v_2, v_4, v_6 are not neighbors of w_1, w_2, thus we get $(v_0, v_4), (v_2, v_6), (w_1, w_2)$ are edges and v_1, w_1, v_3, v_5, w_2, v_7 induced P_3. So we can draw Petersen graph by putting the edge (w_1, w_2) in the center of 8-cycle. For example 8-cycle $\{12, 34, 51, 24, 13, 52, 41, 35\}$ and edge $\{23, 45\}$.

Last, for the 9-cycle induced by $\{v_0, \dots v_8\}$ and the remain vertex as w. May assume v_0 is adjacent to w, by *Corollary 1.1.40* and Petersen graph has no 7-cycle, then the other two neighbor of w must be v_3, v_6 and $(v_1, v_5), (v_2, v_7), (v_4, v_8)$ are edges. So we can draw Petersen graph by putting 45 in the center of 9-cycle. For example 9-cycle $\{12, 34, 51, 23, 41, 52, 13, 24, 35\}$ and 45.





25. Suppose Petersen Graph has a cycle of length 7, may assume this is induced by $a_1, \dots, a_7$ Since a_1 and a_4 are not adjacent, by *Proposition 1.1.38* they has

exactly one comment neighbor, say p. But a_1 and a_5 are also not adjacent, then they still has only one comment neighbor, say q. If $p = q$, then p, a_3 and a_4 induced a 3-cycle, but by *Corollary 1.1.40* Petersen Graph has girth 5, so we have a contradition. If $p \neq q$, then p, q, a_2 and a_7 are neighbors of a_1, thus $deg(a_1) = 4$, a contradiction. Hence Petersen Graph has no cycle of length 7.

Note: When I write $v_1 \ldots, v_k$ induced a cycle C_k, that means v_i is adjacent to v_{i+1} for $i = 1, \ldots, k-1$ and v_1 adjacent to v_k.

26. Fixed a vertex v, let $u_1, \ldots, u_k$ be neighbors of v. Since the graph has girth 4, then $\{u_1, \ldots, u_k\}$ is an independent set. Thus there u_1 has $k-1$ neighbors (say $w_1, \ldots, w_{k-1}$) which are different from $v, u_2, \ldots, u_k$. So there are at least $1 + k + (k-1) = 2k$ vertices.

    If it has exactly $2k$ vertices, thus $w_1, \ldots, w_{k-1}$ must also be the neighbors of $u_2, \ldots u_k$. That means G is biclique $K_{k,k}$, where one partite set is $\{v, w_1, \ldots, w_{k-1}\}$ and the other is $\{u_1, \ldots, u_k\}$.

27. Fixed a vertex v, let $u_1, \ldots, u_k$ be neighbors of v. Since the graph has girth 5, then $\{u_1, \ldots, u_k\}$ is an independent set. Let the $k-1$ neighbors of u_i be $w_{i,1}, \ldots, w_{i,k-1}$. If u_i and u_j has another neighbor different from v, then it has 4-cycle, a contradiction. So there are at least $1+k+k(k-1) = k^2+1$ vertices. For $k = 2$, C_5. For $k = 3$, Petersen Graph.

28. Clearly the graph is simple, then it has no 1-cycle and 2-cycle. A 3-cycle would need three pairwise disjoint k-sets, that means we need at least $3k$ elements. But $3k > 2k + 1$ for $k \geq 3$, thus there is no 3-cycle.

    For each two non-adjacent vertices S_1, S_2, there are at least $k + 1$ elements in $S_1 \bigcup S_2$, then there are at most k elements without choosing, that is S_1 and S_2 has at most one neighbor. So there is no 4-cycle.

    If there exists vertices $A_1, \ldots, A_5$ induced 5-cycle. W.L.O.G, may assume $A_1 =$

$\{1,\ldots,k\}$ and $A_2=\{k+1,\ldots,2k\}$. Thus A_3 must be $\{1,\ldots,k,2k+1\}\backslash\{t_1\}$, A_5 must be $\{k+1,\ldots,2k,2k+1\}\backslash\{t_2\}$. Since each element of A_4 must be choosen from $\{1,2,\ldots,2k+1\}\backslash(A_3\bigcup A_5)=\{t_1,t_2\}$. But A_4 must contain $k>2$ elements, a contradiction.

Consider the 6-cycle $\{1,\ldots,k\}$, $\{k+1,\ldots 2k\}$, $\{1,\ldots,k-1,2k+1\}$, $\{k,\ldots,2k-1\}$ $\{1,\ldots,k-1,2k\}$, $\{k+1,\ldots,2k-1,2k+1\}$, so the girth is 6.

29. Fixed one person, say P, then he must have 3 acquaintances or 3 strangers. If he has 3 acquaintances, say A_1,A_2,A_3, if at least two of A_1,A_2,A_3(say A_1,A_2) are acquaintance, then P,A_1,A_2 are mutual acquaintances. Otherwise, A_1,A_2,A_3 are mutual strangers. Similarly discussion for P has 3 strangers. So we complete the proof.

30. First of all, we know A is symmetric. Since G is a simple graph, then $A_{i,j}=1$ or 0 depend on v_i adjacent to v_j or not. Thus i-diagonal entry in A^2 is $\sum\limits_{j=1}^{n}A_{i,j}^2=\sum\limits_{j=1}^{n}A_{i,j}$, since $A_{i,j}=A_{i,j}^2$. That means i-diagonal entry is the number of neighbors of v_i, thus it is the degree of v_i.

The i-diagonal entry in MM^T is $\sum\limits_{j=1}^{m}M_{i,j}^2=\sum\limits_{j=1}^{m}M_{i,j}$, since $M_{i,j}=M_{i,j}^2$. That means i-diagonal entry is the number of edges with endpoint v_i, thus it is also the degree of v_i.

If $i\neq j$, $A_{i,j}$ is the number of common neighbor of v_i and v_j. Since $A_{i,j}=\sum\limits_{k=1}^{n}A_{i,k}A_{k,j}$, and $A_{i,k}A_{k,j}=1$ if and only if $A_{i,k}=1=A_{k,j}$. That means v_k is adjacent to v_i and v_j.

If $i\neq j$, $MM^T_{i,j}$ is the number of edges with endpoints v_i and v_j. Since $A_{i,j}=\sum\limits_{k=1}^{m}M_{i,k}M^T_{k,j}$, and $M_{i,k}M^T_{k,j}=1$ if and only if $M_{i,k}=1=M^T_{k,j}$. That means v_i and v_j are both endpoints of e_k.

Note: For $i\neq j$, we have $MM^T_{i,j}=A_{i,j}$.

31. *Necessity:* Since $G\cong\overline{G}$, then $e(G)=e(\overline{G})$. But the number of $e(G)+e(\overline{G})=$

$n(n-1)/2$, thus $e(G) = n(n-1)/4$. Hence n or $n-1$ must be divisible by 4.

Sufficiency: Suppose $n = 4m$, let graph G_m has follows:

$$V(G_m) = \{a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m, d_1, \ldots, d_m, \}.$$
$$E(G_m) = \{(a_i, b_j), (b_i, c_j), (c_i, d_j) | \forall i, j\} \bigcup \{(a_i, a_j), (d_i, d_j) | \forall i \neq j\}$$

Let $V(G_m) = V(H)$ and $f : V(G_m) \to V(H)$, where

$$f(a_i) = b_i, f(b_i) = d_i, f(c_i) = a_i, f(d_i) = c_i$$

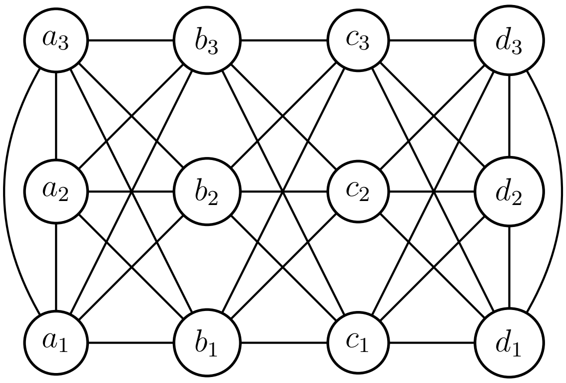
then we have

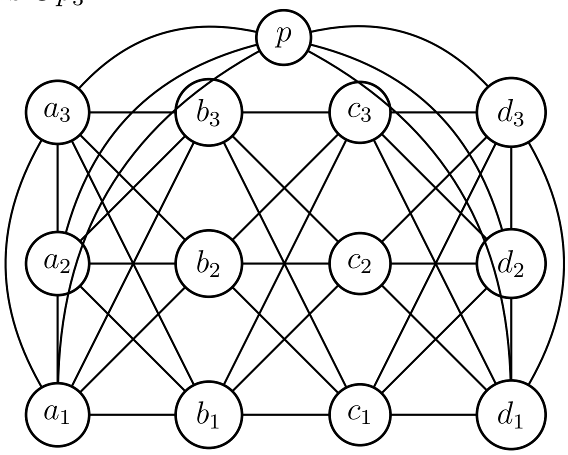
$$E(H) = \{(f(a_i), f(b_j)), (f(b_i), f(c_j)), (f(c_i), f(d_j)) | \forall i, j\}$$
$$\bigcup \{(f(a_i), f(a_j)), (f(d_i), f(d_j)) | \forall i \neq j\}$$
$$\Longrightarrow E(H) = \{(b_i, d_j), (d_i, a_j), (a_i, c_j) | \forall i, j\} \bigcup \{(b_i, b_j), (c_i, c_j) | \forall i \neq j\}$$

It is easy to see that bijection f is isomorphic, then $H \cong \overline{G_m}$.

If $n = 4m + 1$, let graph Gp_m where $V(Gp_m) = V(G_m) + \{p\}$ and $E(Gp_m) = E(G_m) \bigcup \{(p, a_i), (p, d_i) | \forall i\}$. Using the same function f above with $f(p) = p$. It is easy to see that bijection f is isomorphic, then $H \cong \overline{Gp_m}$.

The graph left below is G_3 and right below is Gp_3



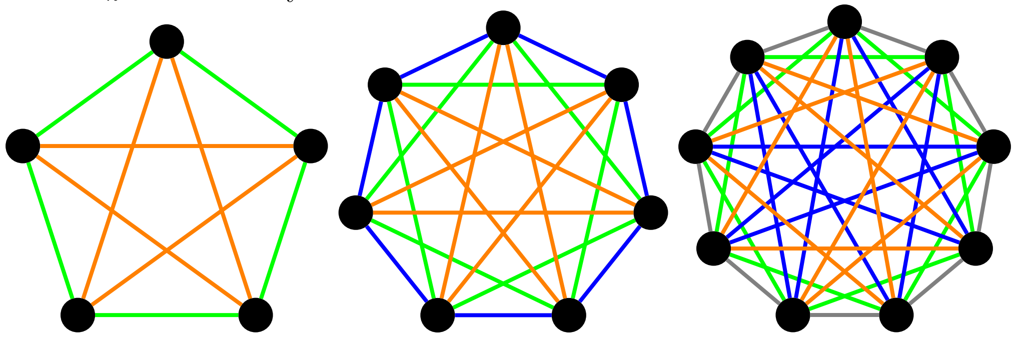


32. If $K_{m,n}$ can decompose into two isomorphic graph G_1 and G_2 if and only if $m \times n$ is even.

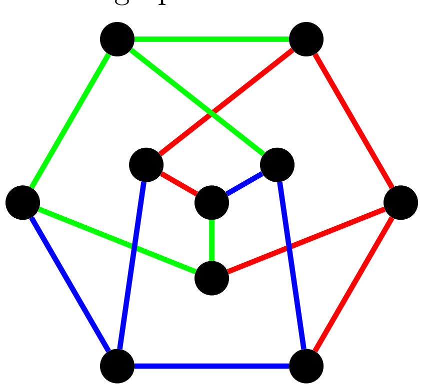
Necessity: Since $e(G_1) = e(G_2)$ and $e(G_1) + e(G_2) = mn$, thus $e(G_1) = mn/2$. Hence $m \times n$ is even.

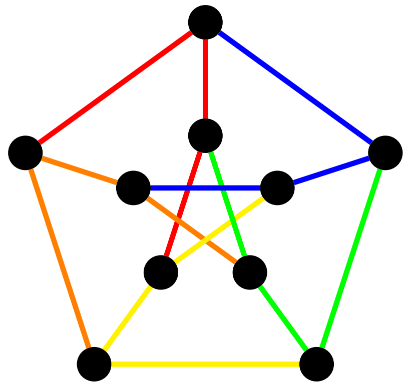
Sufficiency: May assume m is even, then $K_{m,n}$ can decompose into two $K_{m/2,n}$.

33. Each C_n is colored by different colors.



34. I use the middle graph below *Definition 1.1.36* to solve the first question, and the left graph to solve the second.





35. *Necessity:* Let K_n decomposite 3 pairwise isomorphic $G_1 \cong G_2 \cong G_3$, then $|E(G_i)| = |E(K_n)|/3 = n(n-1)/6$ is integer. Since $n(n-1)$ must be even, thus $n(n-1)/3$ must be integer. However only one number of $n+1, n, n-1$ is divisible by 3, thus we have $n+1$ is not divisible by 3.

Sufficiency: Let the vertex of K_n be $p_0, p_1, \ldots, p_{n-1}$, if $n = 3k$, for $i = 0, 1, 2$

$$V(G_i) = \{p_t, p_{t+1} \big| t = 3j + i, 0 \le t < n\}$$

$$E(G_i) = \{(p_t, p_k), (p_t, p_{t+1}) \big| t = 3j + i, k = 3m + i, 0 \le t, k < n\}$$

Thus easily to see $f_i : V(G_i) \to V(G_{i+1})$ where $f_i(p_t) = p_{t+1}$ is an isomorphism.

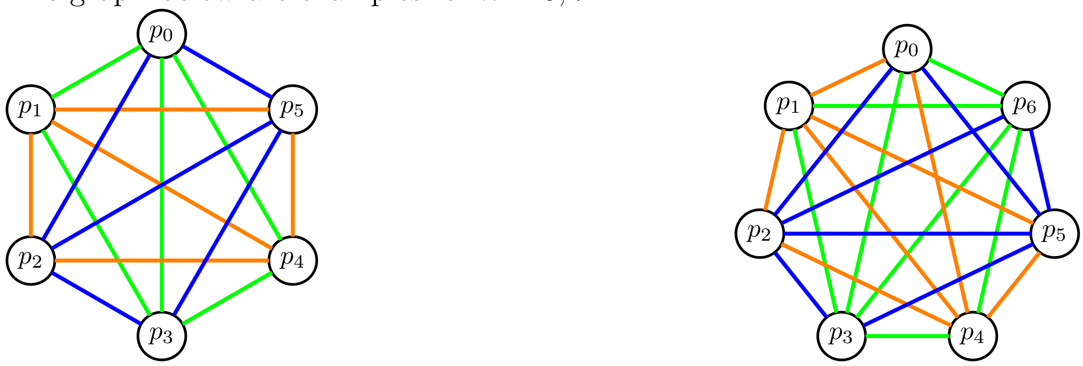
if $n = 3k+1$, for $i = 0, 1, 2$

$$V(G_i) = \{p_t, p_{t+1} \big| t = 3j+i, 0 < t < n\} \cup \{p_0\}$$

$$E(G_i) = \{(p_t, p_k), (p_t, p_{t+1}), (p_0, p_t) \big| t = 3j+i, k = 3m+i, 0 < t, m < n\}$$

Thus easily to see $f_i : V(G_i) \to V(G_{i+1})$ where $f_i(p_t) = p_{t+1}$ for $t > 0$ and $f_i(p_0) = p_0$ is an isomorphism.

The graph below are examples for $n = 6, 7$.



36. If K_n decompose into triangles, then the number of triangles is $E(K_n)/3 = n(n-1)/6$. Hence n must be $6k+1$, $6k+3$, $6k+4$ or $6k$. But each vertex v contributes 2 edges in triangle, then $deg(v) = n-1$ must be even, So $n = 6k+1$ or $6k+3$.

37.

38.

39.

40. It's easy to see the automorphism of P_n, C_n and K_n is 2, $2n$ and $n!$.

41.

42.

43.

44.

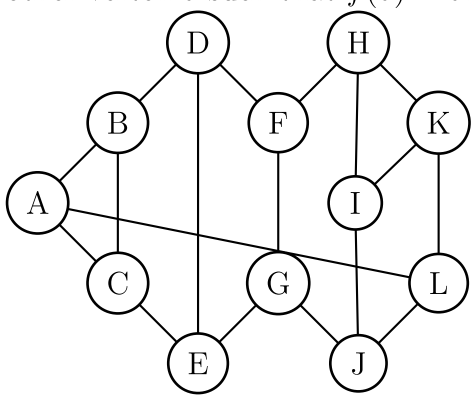
45. First of all, let $Eg(u,v)$ be the minimum length of cycle which contains edge uv and $EG(v) = \{Eg(u,v)|u \in N(v)\}$. It is clearly that if there exists an automorphism f with $f(p) = q$ if and only if $EG(p) = EG(q)$.

Now consider the graph below and given automorphism f. Since the graph below contains only two 3-cycle $\{A, B, C\}$ and $\{H, I, K\}$, we only need to check $EG(A), EG(B), EG(C), EG(H), EG(I), EG(K)$. Thus by carefully counting, we have

$$EG(A) = \{3,6,7\}; EG(B) = \{3,4,7\}; EG(C) = \{3,4,6\}$$

$$EG(H) = \{3,5,6\}; EG(K) = \{3,4,6\}; EG(I) = \{3,4,5\}$$

So $f(A) = A$, $f(B) = B$, $f(H) = H$ and $f(I) = I$ and it is east to check the other vertex v such that $f(v) = v$. Hence f must be identity.



46.

47.