Exercise of Section 1.2

Written by Hsin-Jung, Wu.

1. (a) False. Consider $2k_2$.

   (b) False. Consider P_4.

   (c) True. Use induction on the length ℓ of a closed trail W.

   Basis step : $\ell = 3$. Clearly, a closed trail with length 3 is 3-cycle.

   Induction step : $\ell > 3$. If W has no repeat vertex v, the we are done. Otherwise, the edges E_i and vertices V_i between appearance of v (leaving one copy of v), and the remaining edges and vertices yields closed trails $\{W_i\}$ with length less than W. So by induction hypothesis, we are done.

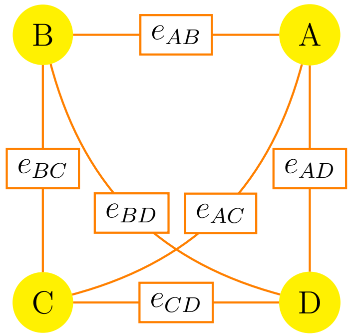
   (d) False. Consider the graph 3-cycle, then the maximal trail is P_2, but 3-cycle is 2-regular.

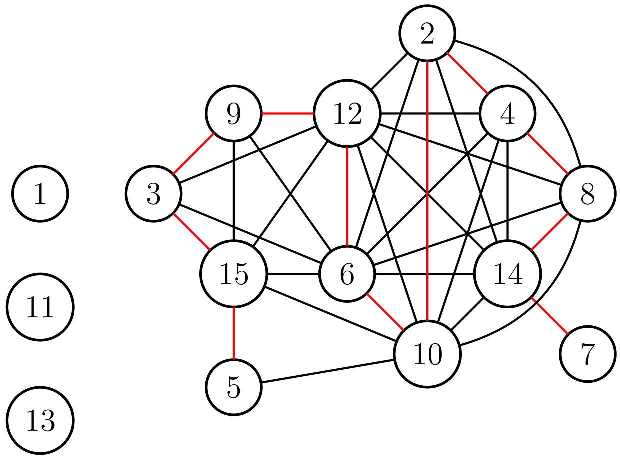
2. Label K_4 as below left.

   (a) K_4 contains a walk that is not a trail. Consider the walk as $A, e_{AB}, B, e_{BC}, C, e_{CD}, D, e_{DA}, A, e_{AC}, C, e_{CB}, B, e_{BD}, D$

   (b) K_4 contains a trail that is not a closed and is not a path. Consider the trail as $A, e_{AB}, B, e_{BC}, C, E_{CD}, D, E_{DA}, A, e_{AC}, C$

   (c) Since each vertex has even degree in a closed trail, and K_4 is 3-regular, thus the closed trail is a connected 2-regular subgraph of K_4. Hence the closed trail must be 3-cycle or 4-cycle.





3. By the graph above right, then we can find G has 4 components and the red line presents the maximum length of a path is 12.

   Generally, consider the vertex set becomes $\{1, 2, \ldots, n\}$, and obviously 1 is isolated vertex. Let G_i be the subgraph induced by $\{p_i, \ldots, \lfloor n/p_i \rfloor p_i\}$ where p_i is prime number and less than n. Thus we have G_i and G_j are connected if and only if $V(G_i) > 1$ and $V(G_j) > 1$. Hence there are $k+2$ components where $k = |\{\lfloor n/p_i \rfloor p_i \big| \lfloor n/p_i \rfloor p_i = 1\}|$, and at least $k+1$ components are isolated vertex.

4. Let A and M be the adjacency and incidence matrix of G, v_t correspond to the tth row and tth column of A and correspond to the ith row of M, $e_k = v_t v_s$ correspond to the kth column of M. Assume $v_t A$ and $v_t M$ be the adjacency and incidence matrix of $G - v_t$ and $e_k A$ and $e_k M$ be the adjacency and incidence matrix of $G - e_k$. Thus

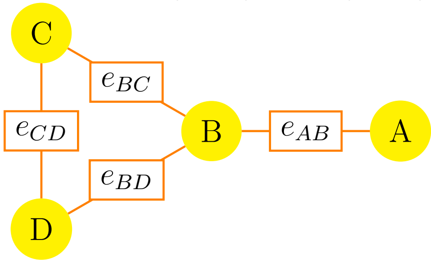
$$v_t A_{i,j} = \begin{cases} A_{i,j} & \text{if } i < t, j < t \\ A_{i+1,j} & \text{if } i \geq t, j < t \\ A_{i,j+1} & \text{if } i < t, j \geq t \\ A_{i+1,j+1} & \text{if } i \geq t, j \geq t \end{cases}$$

   and $v_k M$ be M delete kth row and delete sth column for all v_s adjacent to v_k.

$$e_k A_{i,j} = \begin{cases} 0 & \text{if } i = t, j = s \\ A_{i,j} & \text{otherwise} \end{cases} \quad e_k M_{i,j} = \begin{cases} 0 & \text{if } (i,j) = (s,k), (t,k) \\ M_{i,j} & \text{otherwise} \end{cases}$$

5. Suppose there is a component, say H, has no neighbor of v, then there is no path u, v-path for all u in H. Hence G is not connected, contradiction.

6. The maximal path are $\{A, e_{AB}, B, e_{BC}, C, e_{CD}, D\}$, $\{A, e_{AB}, B, e_{BD}, D, e_{DC}, C\}$ and obviously they are both maximum path. The maximal clique is $\{A, B\}$ and $\{B, C, D\}$ and $\{B, C, D\}$ is also the maximum clique. The maximal independent set is $\{A, C\}$ and $\{A, D\}$ and both they are also the maximum clique.

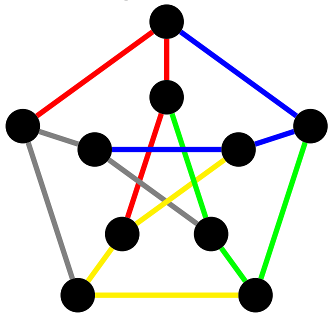


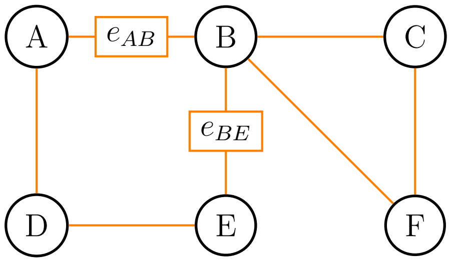
7. *Necessity:* If G is not connected, then say G has components $\{G_i\}$ and all G_i are also bipartite. Assume G_i has bipartition A_i and B_i. Thus $\{A_i\}$ and $\{B_i\}$ is a bipartition, $\{B_1\} \cup \{A_i\} \backslash \{A_1\}$ and $\{A_1\} \cup \{B_i\} \backslash \{B_1\}$ is another bipartition.
Sufficiency: If G is connected, fixed a vertex v and let a set $A = \{u_i\}$ where length of u_i, v-path is odd, and B collects the remaining vertex. If there exists $u, w \in A$ or B such that u, v are adjacent, then u, v-path, v, w-path and edge uv forms a closed odd walk. By *Lemma 1.2.15* and *Theorem 1.2.18* we have G is not bipartite, a contradiction. Hence this is a bipartition of G. If there exists another bipartition of G, assume the bipartite sets are C, D and $v \in C$. If there is $u \in A \cap C (u \in B \cap D)$, that means there exists odd(even) u, v path P. Let P induced by $v = v_0, v_1, \ldots, u = v_{2k+1}(v_{2k})$, then easily we get v_{2i} must in D, that is $v \in D$, a contradiction. Since $V(G) = A \cup B = C \cup D$, so we have $A = C$ and $B = D$.

8. By *Theorem 1.2.16*, every vertex has even degree, then $K_{m,n}$ is Eulerian if and only if m and n are both even.

9. By definition, we have each open(closed) trail has are 2(0) vertices contribute

odd degree and the other vertices contribute even degree. Since Petersen graph has ten vertices and 3-regular, then It can be decompose into at least 5 trail and the graph below is an example which decompose into exactly 5 trail(path).





10. (a) True. By *Exercise 1.2.8*, we have $m = 2p, n = 2q$. Thus the number of edges is $mn/2 = 2pq$.

    (b) False. The graph above right is an counterexample.

11. False. The graph above right is an counterexample. Obviously, there is no Eulerian circuit C such that e_{BE} and e_{AB} appear consecutively.

12.

13. (a) Let w be the neighbor of u, then there exists w, v-walk of length $\ell - 1$, thus there by induction hypothesis there is w, v-path. Hence edge $uw + w, v$-path forms u, v-path.

    (b) Consider shortest u, v-walk P in W, if P is not a path, there must exists repeats vertex t in P. But delete the edges and vertices between appearance of t (leaving one copy of t) will make a shorter path, hence P must be path.

14.

15.

16.

17. Let $v = (1, 2, \ldots, n)$, for every vertex $u_0 = (i_1, i_2, \ldots, i_{k-1}, i_k = 1, i_{k+1}, \ldots, i_n)$, we interchange i_{k-1}, i_k to be $(i_1, i_2, \ldots, i_k = 1, i_{k-1}, i_{k+1}, \ldots, i_{n-1}, i_n)$ thus repeat this process, we can make u become $u_1 = (1, i_1, i_2, \ldots, i_{k-2}, i_{k-1}, i_{k+1}, \ldots, i_{n-1}, i_n)$ That means there is u_0, u_1-path. Now, by the same way as above, we can make u_2 become $u_2 = (1, 2, \ldots,)$. So use the same method, we can make u_0 become v, thus there exists u_0, v-path, that means G_n is connected.

18. Let $v = (\underbrace{0, \ldots, 0}_{k})$ and $u = (1, \underbrace{0, \ldots, 0}_{k-1})$, it is easy to see the vertex w with the same parity of "0" as $v(u)$, thus we have two vertices with the same parity of "0" are connected. But v, u are not connected, the G has exactly two components.

19.

20. Given p, q in $\overline{G} - v$, if p and q are in different components of $G - v$, then p adjacent to q. Otherwise, there exists a vertex r s.t. r and p in the different components, then p, r, q forms P_3. So $\overline{G} - v$ is connected.

21. *Necessity:* Let v be the cut vertex of G, by *Exercise 1.2.20*, we have v is not cut vertex of $\overline{G}$. Suppose every vertex has degree at least 2, thus each component has at least 2 vertices. If we delete any vertex u other than v, for each two vertices $a, b \neq v$,

    - if a, b in the different component of $G - v$, then a, b are adjacent in $\overline{G}$;
    - if they are in the same component K, choose vertex c in the different from K, then a, c, b induced P_3 in $\overline{G}$;
    - if a, v is not adjacent in G, then they are adjacent in $\overline{G}$;
    - if a, v is adjacent in G, choose vertex c in the different component from a in, then a, c, v induced P_3 in $\overline{G}$;

    Hence $\overline{G}$ has no cut vertex, so $\overline{G} \ncong G$, contradiction.

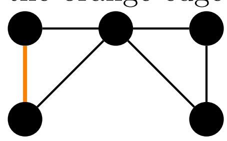
Sufficiency: If G has a vertex u with degree 1, say v is u's neighbor. Thus easily to see v is a cut vertex of G.

22. *Necessity:* If G has an partition A, B and for each two vertices $a_i \in A$ and $b_j \in B$ such that a_i and b_j are not adjacent, then G must be disconnected.

    Sufficiency: If each partition A, B of G, there exist two vertices $a \in A$ and $b \in B$ such that a, b are adjacent, thus we only consider for two vertices $u \in A$ and $v \in B$ has u, v-path formed by u, a-path, edge ab and b, v-path, hence G is connected.

23. (a) True. Since simple connected graph G is not complete graph, then for each vertex v there exists v, u-path P with length 2, that P is P_3.

    (b) False. The graph below is a counterexample, easy to see the orange edge not belong to an induced subgraph isomorphic to P_3.



24. *Remark:* I think we need G must be "connect", otherwise $2K_2$ is a counterexample. So I will prove it under the condition that G is connected.

    Let v be the vertex with the minimum degree and $\{u_i\}$ be neighborhood of v. If there exists another vertex w, then w must be adjacent some u_i, thus w, u_i, v induced P_3. Otherwise u_i must be adjacent to u_j for $i \neq j$. Thus G is complete graph.

25. **induction on number of vertices**: Let n be the number of vertices.

    Basis step : $n = 2$, trivial.

    Induction step: Suppose $n = k - 1$ is hold, then we delete a vertex v of G, then of $G - v$ has $k - 1$ vertices and has no odd cycle. So by induction hypothesis, $G - v$ has partite sets A and B. Let u, w are neighbors of v, since u, v, w cannot induce 3-cycle, then we can put u, w into the same partite set, say A. Thus A and $v \cup B$ are both independent sets, hence G is bipartite graph.

induction on number of edges: Let m be the number of edges.

Basis step : $m = 0$, trivial.

Induction step: Suppose $m = k - 1$ is hold, then we delete a edge uv of G, then $G - vu$ has $k - 1$ edges and has no odd cycle. So by induction hypothesis, $G - uv$ has partite sets A and B. If u, v are in different partite set, then we are done. If u, v both in A (or B) and exist u, v-path, then u, v-path and edge uv induce odd cycle, contradiction. Otherwise u, v both in A (or B) and has no u, v-path, let

$$C = \{t \in V(G) \big| t, v\text{-path is odd or } t, u\text{-path is even.}\}$$

$$D = \{t \in V(G) \big| t, v\text{-path is even or } t, u\text{-path is odd.}\}$$

Thus C and D are both independent sets, hence G is bipartite graph.

26. *Necessity:* If G is bipartite and let A, B be its partite sets. For each supgraph H let $H_A(H_B) = H \cap A(B)$. Thus easy to see that both H_A, H_B independent and either $|V(H_A)|$ or $|V(H_B)|$ not less than $|V(H)|/2$.

    Sufficiency: If G has a subgraph H isomorphic to odd cycle (say length $2k+1$), then easily to see every independent set of H consist at most k vertices, a contradiction. Thus by *Theorem 1.2.18*, G is bipartite.

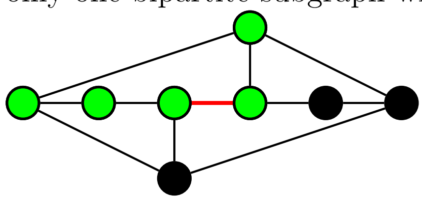
27. We use the hint to prove the following the statement.

    > If two permutation are adjacent, then their inversion has different parity.

    Let $\sigma_1 = (i_1, \ldots, i_{k-1}, i_k = x, i_{k+1}, \ldots, i_{m-1}, i_m = y, i_{m+1}, \ldots i_n)$ and $\sigma_2 = (i_1, \ldots, i_{k-1}, i_m = y, i_{k+1}, \ldots, i_{m-1}, i_k = x, i_{m+1}, \ldots i_n)$ For convenience, say $x < y$ and let ℓ be the number of $\{i_j \big| k < j < m, x < i_j < y\}$. Since switching x, y will increase(decrease) $2\ell+1$ inversions, so we prove the statement above. If G_n contains odd cycle C_{2j+1} induced by $v_1, v_2, \ldots v_{2j+1}$, then by the statement above we have v_1, v_{2j+1} have the same parity of inversions. But v_1, v_{2j+1} are adjacent, contradiction. Thus by *Theorem 1.2.18* we have G_n is bipartite.

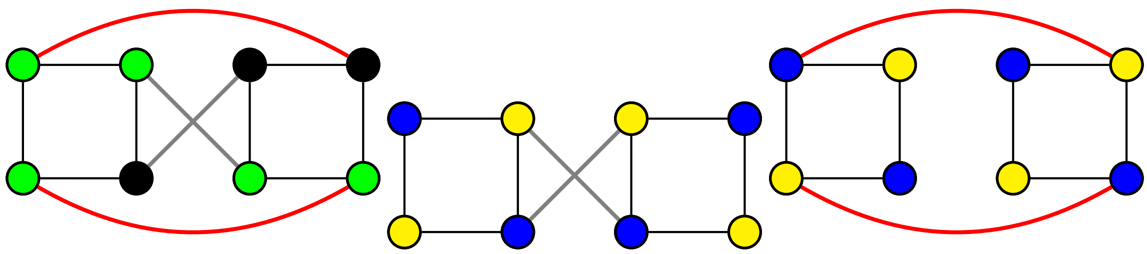
28. Since the green vertices of the graph G left below has induced 5-cycle, thus we must delete at least one edge to make it become bipartite. Thus delete the red edge of G such that it becomes graph G' right below. Since G' is 2-colorable, we have G' is bipartite.

    If we delete other edge e, then $G - e$ must have induced 5-cycle, thus G' is the only one bipartite subgraph with 10 edges.



    Since the green vertices of the graph H left below has induced 5-cycle, thus we must delete at least one edge to make it become bipartite. But it is easy to check for each edge e, $H - e$ still contains induced 5-cycle, hence we need to delete at least two edges. Thus delete the red edges of G such that it becomes graph H' middle below. Since H' is 2-colorable, we have H' is bipartite.

    The graph right below is another one bipartite subgraph with the same edges as H'.



29. If G is not bipartite, then G has odd cycle C, thus C is either C_3 or contains P_4 as an induced subgraph, contradiction. Hence G is bipartite with partite sets A, B. Suppose $a \in A$ and $b \in B$ such that a, b are non-adjacent, since G is connected, there exists a, b-path $P = \{a, x, \ldots, y, b\}$ where $x(y) \in B(A)$ that

$a, x(b, y)$ are adjacent. But P contains P_3 as induced subgraph, contradiction. Hence G must be biclique.

30. Use induction on k.

Basis step: $k = 1$, it's obviously.

Induction step: Suppose $k = r$ holds, for each entry i, j of A^{r+1}

$$A^{r+1}[i, j] = \sum_{k=1}^{n} A^r[i, k] \times A[k, j]$$

Since each v_i, v_j-walk can be decomposed into v_i, v_k-walk and edge v_k, v_j, by induction hypothesis $A^r[i, k]$ is the number of v_i, v_k-walk of length r, then $A^{r+1}[i, j]$ is indeed the number of v_i, v_j-walk of $r + 1$.

$$\begin{aligned} & G \text{ is bipartite} && (1) \\ \Longleftrightarrow & G \text{ has no odd cycle} && (2) \\ \Longleftrightarrow & G \text{ has no odd closed walk} && (3) \\ \Longleftrightarrow & \text{the diagonal entries of } A^r \text{ is all } 0, \text{ for each odd } r && (4) \end{aligned}$$

(1) $\Longleftrightarrow$ (2) by *Theorem 1.2.18.* Since an odd cycle is also an odd walk, and by *Lemma 1.2.15* we have (2) $\Longleftrightarrow$ (3) Use the prove above, we have (3) $\Longleftrightarrow$ (4). Hence we complete the proof.

31.

32. The true statement should be

> Every "maximum" trail in a even graph "has at most one nontrivial component" is an Eulerian circuit.

Use *Theorem 1.2.26* we have the graph G is Eulerian, then G has an Eulerian circuit. But Eulerian circuit is a maximum trail (since it travels all edges), then every maximum trail T must contain all edges of G, thus by definition T is Eulerian circuit.

33. **induction on k:** Let the graph be G and v, u be the vertices with odd degree. Basis step : $k = 1$ we add an edge $e = uv$, thus $G+e$ is even graph. By *Theorem 1.2.26*, G has an Eulerian circuit $C = \{v, \ldots, \ldots, u, e, v\}$, thus $\{v, \ldots, \ldots, u\}$ is a trail contains all edge of G.

    Induction step : Suppose $k = r$ holds. If $k = r + 1$ then we add an edge $e = uv$, thus $G + e$ has $2r$ odd vertices. By induction hypothesis, $G + e$ can be decomposed into r trails, $T_1, \ldots, T_r$ and may assume trail T_1 contains e. But delete an edge of one trail can make it become two trails, thus say $T_1 - e$ become two trails T_{11}, T_{12}. Thus G is decomposed into the $r+1$ trails, $T_{11}, T_{12}, T_2, \ldots, T_r$.

    induction on the number of edges: Let m be the number of edges, the graph be G.

    Basis step : $m = 1$, trivial, since $G = K_2$.

    Induction step : Suppose $m = r$ holds, we consider $m = r + 1$.

    (a) Suppose there exist some vertex v with degree 1 and say u as v's neighbor, thus $G - v$ is connected and has r edges.

        - If $deg_G(u)$ is odd then $G - v$ contains $2(k - 1)$ odd vertices, thus by induction hypothesis, $G - v$ can be decomposed into $k - 1$ trails $T_1, \ldots, T_{k-1}$, thus G can be decomposed into k trails, $T_1, \ldots, T_{k-1}$ and edge uv.
        - If $deg_G(u)$ is even then $G - v$ contains $2k$ odd vertices, thus by induction hypothesis, $G - v$ can be decomposed into k trails $T_1, \ldots, T_k$. Since $deg_{G-v}(u)$ is odd, thus u must be an endpoint of some trails. May assume u is an endpoint of T_1, Thus add edge uv to T_1, then G can be decomposed into k trails, $T_1 \cup \{uv\}, T_2, \ldots, T_k$.

    (b) Suppose each vertex has degree at least 2, by *Lemma1.2.25*, G has a cycle C. Since G is connected and has odd vertices, there exists a vertex v in C

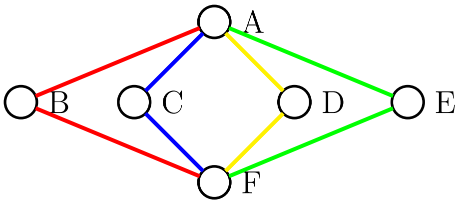
with degree at least 3. By *Theorem 1.2.14* edge $e = uv$ of this cycle is not cut-edge then $G - e$ is connected and has r edges.

- If both u, v are even vertex in G, then they are both odd in $G-e$, thus $G-e$ has $2(k+1)$ odd vertices. By induction hypothesis, $G-e$ can be decomposed into $k+1$ trails $T_1, \ldots, T_{k+1}$. Since u, v are odd vertex in $G-e$, then u, v must be endpoint of some trails. If u, v are endpoints of distinct trails, say T_1, T_2, thus $T_1 \cup T_2 \cup e$ is a trail. Hence G can be decomposed into k trails $T_1 \cup T_2 \cup e, T_3, \ldots, T_{k+1}$. Otherwise u, v must be endpoints of the same trail, say T_1. Since $deg_{G-e}(v) \geq 2$, then v must be internal vertex of some T_i, say T_2. Thus extend T_2 be T_2' by replacing v, u-tail, T_1, and u, e, v with v. then G can be decomposed into k trails, $T_2', T_3, \ldots, T_{k+1}$.
- If both u, v are odd vertex in G, then they are both even in $G-e$, thus $G-e$ has $2(k-1)$ odd vertices. By induction hypothesis, $G-e$ can be decomposed into $k-1$ trails $T_1, \ldots, T_{k-1}$, thus G can be decomposed into k trails, $T_1, \ldots, T_{k-1}$ and edge uv.
- Otherwise, may assume u is even, then $G-e$ has $2k$ odd vertices, by induction hypothesis, $G-v$ can be decomposed into k trails $T_1, \ldots, T_k$. Since $deg_{G-e}(u)$ is odd, thus u must be an endpoint of some trails. May assume u is an endpoint of T_1, Thus add edge uv to T_1, then G can be decomposed into k trails, $T_1 \cup \{uv\}, T_2, \ldots, T_k$.

Yes, it's true. Let G has components G_i and each G_i has $2k_i$ odd vertices, thus by prove above we have each G_i can be decomposed into k_i trails. Thus a graph G has $2\sum k_i$ odd vertices can be decomposed into $\sum k_i$ trails.

34. Since B, C, D, E has degree 2, then the edges with the same color of any Eulerian circuit must appear consecutively. Then it is equivalent to count the ring

permutation of 4 colors, so we have $4!/4 = 6$ equivalent classes.



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37. Given u, v-path $P = \{u = p_0, p_1, \ldots, p_n = v\}$ and v, w-path $Q = \{v = q_0, q_1, \ldots, q_m = w\}$. Thus there exists u, v-walk $W = \{u = p_0, p_1, \ldots, p_n = v = q_0, q_1, \ldots, q_m = w\}$. Hence by *Lemma 1.2.5* there exists a u, w-path.

38. May assume G is connected and suppose there exists a smallest n such that n-vertex graph G with at least n edges but G has no cycle. That is for each $m < n$, m-vertex graph with at least m edges contains cycle. By *Theorem 1.2.14* each edge e is a cut-edge of G, thus $G - e$ has 2 components G_1, G_2, n vertices and at least $n - 1$ edges. Clearly, G_i has no cycle. Let $|V(G_i)| = n_i$, if each G_i has at most $n_i - 1$ edges, then $G - e$ has at most $n_1 - 1 + n_2 - 1 = n - 2 < n - 1$ edges, contradiction. Thus may assume G_1 has at least n_1 edges, thus G_1 has cycle, contradiction. Hence G must have cycle.

39. Consider the maximal path $P = \{p_1, \ldots p_n\}$, thus all p_1's neighbor must be in P, since P is not extendible. Thus assume p_k and p_m are neighbors of p_1, where $k < m$. If $p_1, \ldots, p_k, p_1$ is not even cycle, then either $p_1, \ldots, p_m, p_{m-1}, \ldots, p_k, p_{k-1}, \ldots, p_1$ or $p_1, \ldots, p_m, p_{m-1}, \ldots, p_k, p_1$ induced a even cycle.

40. Let $P(Q)$ consist $\{p_1, \ldots, p_n\}(\{q_1, \ldots, q_n\})$. Since G is connected and suppose P, Q has no common vertex, then we have there exists p_k, q_m-path R such that there is no p_i, q_j-path where $k < i$ or $k = i, m < j$.

- if $k < n/2$ and $m < n/2$, thus $p_n, \ldots, p_k$, R and $q_m, \ldots, q_n$ induced a path of length at least $n - k + +1 + n - m > n$, contradiction.
- if $k < n/2$ and $m \geq n/2$, thus $p_n, \ldots, p_k$, R and $q_m, \ldots, q_1$ induced a path of length at least $n - k + 1 + m > n$, contradiction.
- if $k \geq n/2$ and $m < n/2$, thus $p_1, \ldots, p_k$, R and $q_m, \ldots, q_n$ induced a path of length at least $k + n + 1 - m > n$, contradiction.
- if $k \geq n/2$ and $m \geq n/2$, thus $p_1, \ldots, p_k$, R and $q_m, \ldots, q_n$, $p_n, \ldots, p_k$, R and $q_m, \ldots, q_1$ induced a path of length at least $k+1+n-m$, $n-k+1+m$. However one of $k + 1 + n - m$, $n - k + 1 + m$ must be greater than n, contradiction.

Hence P, Q must have a common vertex.

41. Consider the longest path $P = \{p_1, \ldots p_n\}$, thus all p_1's neighbor must be in P, since P is not extendible. If there exists vertex $u \neq p_1, p_3$ which is p_2's neighbor, then u's neighbor must be in P, since P is not extendible. Thus $G - p_1, u$ is still connected and p_1, u has common neighbor p_2. If there is such u, then all p_2's neighbor must be in P. Thus $G - p_1, p_2$ is still connected and p_1, p_2 are adjacent.

42. Choose a vertex of maximum degree k, say v, and let $u_1, \ldots, u_k$ be v's neighbors. Suppose there is another vertex w, since G is connected, then may assume w is adjacent to v_1. For each $j > 1$, if u_j is not adjacent to u_1, then w, u_1, v, u_j induce P_4 or C_4. Hence u_1 has $k + 1$ neighbors $\{v, u_2, \ldots, u_k, w\}$, a contradiction. So v must be adjacent to all other vertices.

43. Use induction on the number k of edges of connected simple graph G.

    Basis step : $k = 2$. Obviously, a connected graph having 2 edges is P_3.

    Induction step : $k > 2$. Assume the claim is true for graph with even edges less than G. Since delete an edge will increase at most one component, thus we delete edge uv of G. If $G - uv$ is still connected, then we delete an edge vw. If $G - uv$ is disconnected, since $G - uv$ has odd edges, then there is a component with odd edges. May assume v in this component with odd edges, then we delete an edge vw. Thus weather $G - uv - vw$ is connected or not, each component of $G - uv - vw$ has even edges. So by induction hypothesis, each component of $G - uv - vw$ can be decomposed into P_3 and u, v, w induced P_3. Hence we are done.

No, consider the graph $2K_2$.